A Linear First-Order Differential Equation Application

A 600-gal tank is filled with 300 gal of pure water. A spigot is opened above the tank and a salt solution containing 1.5 lb of salt per gallon of solution begins flowing into the tank at a rate of 3 gal/min. Simultaneously, a drain is opened at the bottom of the tank allowing the solution to leave the tank at a rate of 1 gal/min.

What will be the salt content in the tank at the precise moment that the volume of solution in the tank is equal to the tank's capacity (600 gal) ?

Solution: Let $x(t)$ = the # of lbs of salt in the tank at time t minutes. $x_0 = x(0) = 0$ lbs.

Let $V(t)$ = the volume of solution in the tank at time t minutes. $V(0) = 300$ gal.

Since solution enters the tank at a rate of 3 gal/min and solution leaves the tank at a rate of 1 gal/min, the volume of solution in the tank is increasing at a constant rate of 2 gal/min.

So, $V(t) = 300 + 2t$ gallons at time t minutes.

Thus, at any time t minutes, the concentration of salt in the solution in the tank is

$$\frac{x(t)}{V(t)} = \frac{x(t)}{300+2t} \text{ lbs of salt/gallon.}$$

let r_{IN} = the rate at which salt is entering the tank at time t

r_{OUT} = the rate a which salt is leaving the tank at time t.

$\frac{dx}{dt}$ = the rate at which the amount of salt is changing at time t.

The rates r_{IN}, r_{OUT}, and $\frac{dx}{dt}$ have units of lbs/min and

$$\frac{dx}{dt} = r_{IN} - r_{OUT}.$$

$$r_{IN} = (1.5 \text{ lbs of salt/gallon}) \times (3 \text{ gal/min}) = 4.5 \text{ lbs/min}$$

$$r_{OUT} = \left(\frac{x(t)}{V(t)} \text{ lbs of salt/gal}\right) \times (1 \text{ gal/min})$$

$$r_{OUT} = \left(\frac{x(t)}{300+2t} \text{ lbs/gal}\right) \times (1 \text{ gal/min}) =$$

$$r_{OUT} = \left(\frac{x(t)}{300+2t} \text{ lbs/min}\right)$$

$$\boxed{\frac{dx}{dt} = 4.5 - \frac{x}{300+2t} \text{ lbs/min}}$$

WITH INITIAL CONDITION $x(0) = 0$

This is a Linear First-order differential equation with $x = x(t)$ for a solution.

It was shown in class that the solution of this equation with $x(0) = 0$ is the function

$$\boxed{x(t) = 450 + 3t - 4500\sqrt{3}\,(300+2t)^{-\frac{1}{2}} \text{ lbs of salt.}}$$

[The derivation of this solution appears at the end of this handout]

Let t_0 = the time precisely when the tank is filled to capacity;

That is $V(t_0) = 600$ gallons.

$V(t_0) = 300 + 2t_0 = 600$

$2t_0 = 300 \Rightarrow \underline{t_0 = 150 \text{ minutes}}$

at time $t_0 = 150$ minutes,

$$x(t_0) = x(150) = 450 + 3(150) - 4500\sqrt{3}\left(300 + 2(150)\right)^{-1/2}$$

$$= 450 + 450 - 4500\sqrt{3}\,(600)^{-1/2}$$

$$= 900 - \frac{4500\sqrt{3}}{\sqrt{600}} = 900 - \frac{4500\sqrt{3}}{\sqrt{3}\sqrt{2}\sqrt{100}}$$

$$= 900 - \frac{4500}{10\sqrt{2}} = 900 - \frac{450}{\sqrt{2}} = 581.8$$

$x(t_0) = 581.8 = 582$ Rounded to the nearest pound.

$\boxed{\text{There are 582 lbs of salt in the tank when the volume of solution in the tank equals the tank's capacity.}}$

DERIVATION OF THE SOLUTION FOR $x(t)$

$$\frac{dx}{dt} = 4.5 - \frac{x}{300 + 2t} \quad \text{and} \quad x(0) = 0.$$

$$\frac{dx}{dt} + \left(\frac{1}{300 + 2t}\right) x = 4.5 = \frac{9}{2}$$

$\dfrac{dx}{dt} + P(t)\, x = Q(t)$ is a Linear First-order D.E. with Integrating Factor $e^{\int P(x)dx}$.

$$\int P(x)dx = \int \frac{1}{300+2t}\,dt = \frac{1}{2}\int \frac{1}{u}\,du \quad \left(\begin{matrix}\text{where } u = 300 + 2t \\ \text{and } du = 2\,dt\end{matrix}\right)$$

$$= \frac{1}{2}\ln u = \frac{1}{2}\ln(300 + 2t) = \ln\left((300 + 2t)^{1/2}\right)$$

$$\therefore I(x) = e^{\int P(x)dx} = e^{\int \frac{1}{300+2t}dt} = e^{\ln\left((300+2t)^{1/2}\right)} = (300 + 2t)^{1/2} = e^{\int P(x)dx}$$

To : $\frac{dx}{dt} + \left(\frac{1}{300+2t} \right) x = \frac{9}{2}$, we multiply

to both sides the integrating factor $(300+2t)^{1/2}$,

to get : $(300+2t)^{1/2} \frac{dx}{dt} + (300+2t)^{1/2} (300+2t)^{-1} x = \frac{9}{2} (300+2t)^{1/2}$

$(300+2t)^{1/2} \frac{dx}{dt} + (300+2t)^{-1/2} x = \frac{9}{2} (300+2t)^{1/2}$

$\therefore \left[(300+2t)^{\frac{1}{2}} \cdot x \right]' = \frac{9}{2} (300+2t)^{1/2}$

$\therefore (300+2t)^{1/2} \cdot x = \int \left[(300+2t)^{\frac{1}{2}} \cdot x \right]' dx = \int \frac{9}{2} (300+2t)^{1/2} dt$

$= \left(\frac{9}{2}\right) \left(\frac{1}{2}\right) \int u^{1/2} du$ (where $u = (300+2t)$ and $du = 2dt$)

$= \frac{9}{4} \times \frac{2}{3} u^{3/2} + C$

$(300+2t)^{1/2} \cdot x = \frac{3}{2} (300+2t)^{3/2} + C$

(multiplying both sides by $(300+2t)^{-1/2}$):

$x = \frac{3}{2} (300+2t) + C (300+2t)^{-1/2} = \underline{450 + 3t + C (300+2t)^{-1/2}}$

when $t=0$, $x=0$, and $0 = \frac{3}{2}(300) + C (300)^{-1/2}$

$0 = 450 + \frac{C}{\sqrt{300}} \Rightarrow \frac{C}{10\sqrt{3}} = -450$

$\boxed{C = -4500\sqrt{3}}$

So, $\underline{x(t) = 450 + 3t - 4500\sqrt{3} (300+2t)^{-1/2}}$ lbs of salt.